Comparison of field theory models of interest rates with market data

Belal E. Baaquie* and Marakani Srikant†
Department of Physics, National University of Singapore, 2 Science Drive, Singapore 117542
(Received 25 September 2002; revised manuscript received 13 May 2003; published 31 March 2004)

We calibrate and test various variants of field theory models of the interest rate with data from Eurodollar futures. Models based on psychological factors are seen to provide the best fit to the market. We make a model independent determination of the volatility function of the forward rates from market data.

DOI: 10.1103/PhysRevE.69.036129 PACS number(s): 89.65.Gh, 05.10.Gg, 05.40.Jc

I. INTRODUCTION

In this paper, we compare field theory models of interest rates with market data, and propose certain modified models inspired from theoretical considerations and observed facts about the interest rates. The fundamental quantity that is modeled is the forward rate $f(t,x)$, which is the interest rate—fixed at time t—for an instantaneous deposit at some time $x>t$ in the future.

The theoretical framework for all these models in Baaquie's formulation [1–3] of forward rates as a two-dimensional quantum field theory. The Baaquie model is a generalization of the Heath-Jarrow-Morton (HJM) model; the key feature of the field theory model is that the forward rates $f(t,x)$ are imperfectly correlated in the maturity direction $x>t$, and which is specified by a rigidity parameter μ. The models we study are the following: (a) forward rates with constant rigidity [1], (b) forward rates with the variation of the spot rate constrained by a parameter, (c) forward rates with maturity dependent rigidity $\mu(x-t)$, and (d) forward rates with nontrivial dependence on maturity specified by an arbitrary function $z=z(x-t)$.

We first briefly review Baaquie's field theory model and review the market data used in this study. We then introduce two variants of Baaquie's model and test these models. We find that the observed correlation structure can be explained by a relatively straightforward two-parameter field theory model that also has a meaningful theoretical interpretation.

II. THE HJM MODEL

Definition of the model

In the HJM model, the forward rates are given by

$$f(t,x)=f(t_0,x)+\int_{t_0}^{t}dt'\,\alpha(t',x)+\sum_{i=1}^{K}\int_{t_0}^{t}dt'\,\sigma_i(t',x)\,dW_i(t'), \tag{1}$$

where W_i are independent Wiener processes. We can also write this as

$$\frac{\partial f(t,x)}{\partial t}=\alpha(t,x)+\sum_{i=1}^{K}\sigma_i(t,x)\,\eta_i(t), \tag{2}$$

where η_i represent independent white noises. The action functional is

$$S[W]=-\frac{1}{2}\sum_{i=1}^{K}\int dt\,\eta_i^2(t). \tag{3}$$

We can use this action to calculate the generating functional, which is

$$Z[j,t_1,t_2]=\int \mathcal{D}W\exp\left[\sum_{i=1}^{K}\int_{t_1}^{t_2}dt\,j_i(t)W_i(t)\right]e^{S_0[W,t_1,t_2]}=\exp\left[\sum_{i=1}^{K}\int_{t_1}^{t_2}dt\,j_i^2(t)\right]. \tag{4}$$

III. FIELD THEORY MODEL WITH CONSTANT RIGIDITY

We now review Baaquie's field theory model presented in Ref. [1] with constant rigidity. Baaquie proposed that the forward rates being driven by white noise processes in Eq. (2) be replaced by considering the forward rates to be a quantum field. To simplify notation, we write the evolution equation in terms of the velocity quantum field $A(t,x)$, and which yields

$$\frac{\partial f(t,x)}{\partial t}=\alpha(t,x)+\sum_{i=1}^{K}\sigma_i(t,x)A_i(t,x) \tag{5}$$

or

$$f(t,x)=f(t_0,x)+\int_{t_0}^{t}dt'\,\alpha(t',x)+\sum_{i=1}^{K}\int_{t_0}^{t}dt'\,\sigma_i(t',x)A_i(t',x). \tag{6}$$

The main extension to HJM is that A depends on x as well as t unlike W which only depends on t.

While we can put in many fields A_i, we will see from our analysis that the generality brought into the process due to the extra argument x will make one field sufficient. Hence, in future, we will drop the subscript for A.

*Electronic address: phybeb@nus.edu.sg
†Electronic address: srikant@srikant.org

Baaquie further proposed that the field A has the free-(Gaussian) free-field action functional

$$S=-\frac{1}{2}\int_{t_0}^{\infty}dt\int_{t}^{t+T_{\mathrm{FR}}}dx\left[A^2+\frac{1}{\mu^2}\left(\frac{\partial A}{\partial x}\right)^2\right] \tag{7}$$

with Neumann boundary conditions imposed at $x=t$ and $x=t+T_{\mathrm{FR}}$. This makes the action equivalent (after an integration by parts where the surface term vanishes) to

$$S=-\frac{1}{2}\int_{t_0}^{\infty}dt\int_{t}^{t+T_{\mathrm{FR}}}dx\,A(t,x)\left(1-\frac{1}{\mu^2}\frac{\partial^2}{\partial x^2}\right)A(t,x). \tag{8}$$

This action has the partition function

$$Z[j]=\exp\left(\int_0^{t_1}dt\int_t^{t+T_{\mathrm{FR}}}dx\,dx'\,j(t,x)\times D(x-t,x'-t)j(t,x')\right) \tag{9}$$

with

$$D(\theta,\theta';T_{\mathrm{FR}})=\mu\frac{\cosh\mu(T_{\mathrm{FR}}-|\theta-\theta'|)+\cosh\mu(T_{\mathrm{FR}}-(\theta+\theta'))}{2\sinh\mu T_{\mathrm{FR}}}=D(\theta',\theta;T_{\mathrm{FR}})\quad\text{(symmetric function of }\theta,\theta'\text{)}, \tag{10}$$

where $\theta=x-t$ and $\theta'=x'-t$. We can calculate expectations and correlations using this partition function. Note that due to the Neumann boundary conditions, the inverse of the differential operator D actually depends on only the difference $x-t$. The above action represents a Gaussian random field with covariance structure D. In Ref. [1], a different form was found as the boundary conditions used were Dirichlet boundary conditions with the end points integrated over. This boundary condition is in fact equivalent to the Neumann condition, which leads to the much simpler propagator above. In the limit $T_{\mathrm{FR}}\to\infty$, which we will usually take, the propagator takes the simple form $\mu e^{-\mu\theta_>}\cosh\mu\theta_<$, where $\theta_>$ and $\theta_<$ stand for $\max(\theta,\theta')$ and $\min(\theta,\theta')$ respectively.

When $\mu\to 0$, this model should go over to the HJM model. This is indeed the case, since $\lim_{\mu\to 0}D(\theta,\theta';T_{\mathrm{FR}})=1/T_{\mathrm{FR}}$. The extra factor of T_{FR} is irrelevant as it is due to the freedom we have in scaling σ and D. The σ we use for the different models are only comparable after D is normalized.[1] After this choice of normalization, the propagator for both the HJM model and the field theory model in the limit $\mu\to 0$ goes as $D(\theta,\theta')\to 1$, showing that the two models are equivalent in this limit.

[1]This freedom exists since we can always make the transformation $\sigma(\theta)\sim\zeta(\theta)\sigma(\theta)$ and $D(\theta,\theta')\sim D(\theta,\theta')/\zeta(\theta)\zeta(\theta')$—where $\zeta(\theta)$ is an arbitrary scale factor—without affecting any result.

The basic model with constant rigidity can be generalized in many different ways. The generalizations to positive valued forward rates and to models with stochastic volatility are studied in Refs. [2,3]. In this paper, we generalize the free-field model to more complex dependence of μ and $f(t,x)$ on the maturity direction θ.

IV. EURODOLLAR MARKET DATA

We use the Eurodollar futures data for the analysis of this paper. A Eurodollar futures contract represents the interest rate on a deposit of US $1 000 000 for three months at some time in the future. At present, futures contracts for deposits of up to ten years into the future are actively traded. Significant historical data for contracts on deposits up to seven years into the future are available.

It was assumed for simplicity that the Eurodollar futures prices directly reflect the forward rate, an assumption previously used in the literature [4]. This is a reasonable assumption as the forward rates are small enough, so that the difference between the logarithmic measure of the forward rate used in theory and the arithmetic rates used in the market are insignificant. If one makes the reasonable approximation that $f(t,\theta)$ is linear for θ between contract times that are separated by a three month's interval, one can use these data as a direct measure of the forward rates.

We have also analyzed treasury bond tick data from the GovPx database, but found it impossible to obtain forward rates accurate enough for our purposes. The main reason for this is that while we were able to obtain reasonably accurate yields for a few maturities, the differentiation required to get the forward rates from the yields introduced too many inaccuracies. This is somewhat unfortunate since treasury bonds represent risk-free instruments, while a small credit risk exists for Eurodollar deposits.

For the following analysis, we used the closing prices for the Eurodollar futures contracts for the period of 1990–1996. This is the same data as used by Matacz and Bouchaud [4], where the spread of the forward rates and the eigenfunctions of its changes in time are analyzed. For our purposes, we found it more useful to look at the scaled multivariate cumulants of the changes in forward rates for different maturity times.

V. ASSUMPTIONS BEHIND THE TESTS OF THE MODELS

The main assumption that is made for all the tests of the models is that of time translation invariance. In other words, we have assumed that $\sigma(t,\theta)$ depends only on θ and not explicitly on t. We also assume that the propagator $D(\theta,\theta')$ has no explicit time dependence, which is possible in principle. It is reasonable and conceptually economical to assume that different times in the future are equivalent. Further, carrying out any meaningful analysis while these quantities are subject to changes in time is more difficult.

Another important assumption that has to be made is that the forward rate curve is reasonably smooth at small intervals at any given point in time. This assumption is very difficult to test in any meaningful sense, given the relative

paucity of data as forward rate data is available only at three-month intervals (which is what necessitates this assumption in the first place). However, the assumption is a reasonable one to make as one would intuitively expect that the forward rate, say, three years into the future would not be too different from that of three years and one month into the future.

In fact, we will show that there seems to be strong evidence of very long term correlations in the movements of the forward rate. This seems to make the smoothness assumption reasonable as the nearby forward rates tend to move together (except possibly at points very close to the current time). This assumption is required as the forward rate data are provided for constant maturity, which we have been denoting by x, while we want data for constant θ, as shown in Fig. 1. With this assumption, we can get the data by a simple linear interpolation. The loss in accuracy due to this linear interpolation is not all that serious if ϵ, the time interval of t between specifications of the forward rates, is small, since the random changes which we are interested in will be much larger than the introduced errors. This same procedure was used in Matacz and Bouchaud [4].

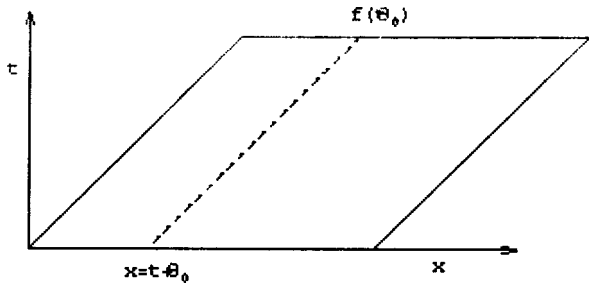

FIG. 1. The lines of constant θ for which we have obtained the forward rates by linear interpolation from the actual forward rates which are specified at constant x.

VI. THE CORRELATION STRUCTURE OF THE FORWARD RATES

An important quantity to look at in the analysis of forward rates $f(t,\theta)$ is the correlation (or scaled covariance) among their changes for different θ. Specifically, we are interested in the correlation between $\delta f(t,\theta)$ and $\delta f(t,\theta')$, where $\delta f(t,\theta)=f(t+\epsilon,\theta)-f(t,\theta)$:

$$\mathcal{C}(\theta,\theta')=\frac{\langle \delta f(t,\theta)\,\delta f(t,\theta')\rangle-\langle \delta f(t,\theta)\rangle\langle \delta f(t,\theta')\rangle}{\sqrt{\langle \delta f^2(t,\theta)\rangle-\langle \delta f(t,\theta)\rangle^2}\sqrt{\langle \delta f^2(t,\theta')\rangle-\langle \delta f(t,\theta')\rangle^2}}. \tag{11}$$

Using a free (Gaussian) quantum field theory (QFT) model, this quantity should be equal to

$$\mathcal{C}_{\mathrm{QFT}}(\theta,\theta')=\frac{D(\theta,\theta')}{\sqrt{D(\theta,\theta)D(\theta',\theta')}}. \tag{12}$$

If we have a model for the propagator $D(\theta,\theta')$, we have a prediction for this correlation structure. Alternatively, we can use the correlation structure to fit free parameters in $D(\theta,\theta')$.

It should be noted that for free (Gaussian) quantum fields the normalized correlation is *independent* of $\sigma(\theta)$, so no assumption of its form has to be made. This is the reason why we used the scaled covariance rather than the covariance itself to perform the study. It is equivalent to fixing the inherent freedom in the quantities σ and D so as to make $D(\theta,\theta)=1$. We then have

$$\sigma(\theta)=\sqrt{\langle \delta f^2(t,\theta)\rangle-\langle \delta f(t,\theta)\rangle^2};\quad D(\theta,\theta)=1. \tag{13}$$

The reduction in the freedom of σ allows us to directly estimate it from data and is shown in Fig. 2. Further, with this normalization for the propagator the correlation between the changes in the forward curve is given exactly by D. The correlation structure in the market estimated from the Eurodollar futures data is shown in Fig. 3. The structure is fairly stable in the sense that the correlation structure for different sections of the data are reasonably similar.

Since the propagator is always symmetric, for purposes of comparison for the different models it will be convenient to calculate only $D(\theta_<,\theta_>)$.

For the one-factor HJM model, this correlation structure is constant as all the changes in the forward rates are perfectly correlated. In other words, $D_{\mathrm{HJM}}(\theta,\theta')=1$. For the two-factor HJM model, the predicted correlation structure is given by

$$\mathcal{C}_{\mathrm{HJM}}(\theta,\theta')=\frac{\sigma_1(\theta)\sigma_1(\theta')+\sigma_2(\theta)\sigma_2(\theta')}{\sqrt{\sigma_1^2(\theta)+\sigma_2^2(\theta)}\sqrt{\sigma_1^2(\theta')+\sigma_2^2(\theta')}}$$

$$=\frac{1+g(\theta)g(\theta')}{\sqrt{1+g^2(\theta)}\sqrt{1+g^2(\theta')}}. \tag{14}$$

We see that this correlation structure depends on a function of $g(\theta)=\sigma_1(\theta)/\sigma_2(\theta)$. Hence, a whole function has to be fitted from the correlation structure, something which is quite infeasible. The covariance given by

$$C(\theta,\theta')=\langle \delta f(t,\theta)\,\delta f(t,\theta')\rangle-\langle \delta f(t,\theta)\rangle\langle \delta f(t,\theta')\rangle \tag{15}$$

might be a better quantity for testing the HJM model, since it is simpler than the normalized correlation. For the two-factor HJM model, the prediction of the covariance has a simpler form

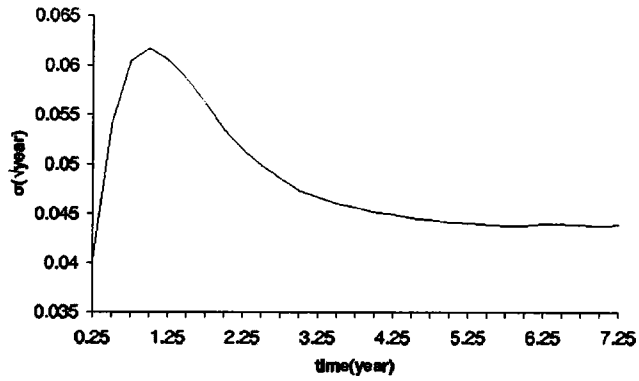

FIG. 2. The empirically determined volatility function for Gaussian field theory models given by $\sigma(\theta) = \sqrt{\langle \delta f^2(t,\theta)\rangle - \langle \delta f(t,\theta)\rangle^2}$, with normalization chosen to be $D(\theta,\theta)=1$.

$$C_{\mathrm{HJM}}(\theta,\theta') = \sigma_1(\theta)\sigma_1(\theta') + \sigma_2(\theta)\sigma_2(\theta'). \tag{16}$$

We still need to specify a functional form for σ_1 and σ_2 as it is not possible to estimate entire functions from data. The usual specification of $\sigma_1(\theta)=\sigma_0$ and $\sigma_2(\theta)=\sigma_1 e^{-\lambda\theta}$ is easily seen to be unable to explain many features of the covariance shown in Fig. 4 such as the peak at one year or the sharp reduction in covariance as the maturity goes to zero.

We hence conclude that the one-factor HJM model is insufficient to characterize the data, while the two-factor HJM model provides us with too much freedom, because we need an entire arbitrary function to explain the correlation structure. If we try to reduce the freedom by theoretical considerations, we are again unable to explain the data.

We will see that the field theory model with constant rigidity, while explaining some features of the correlation, does not predict the correlation very well. We are hence led to consider various generalizations of the constant rigidity model.

VII. ANALYSIS OF FIELD THEORY MODEL WITH CONSTANT RIGIDITY

We have analyzed this model in detail in the preceding section. We have seen that the model describes the changes in the forward rates in terms of a Gaussian random field A whose structure is defined by the action in Eq. (7). For convenience, we repeat the action below in terms of the variables t and $\theta = x - t$,

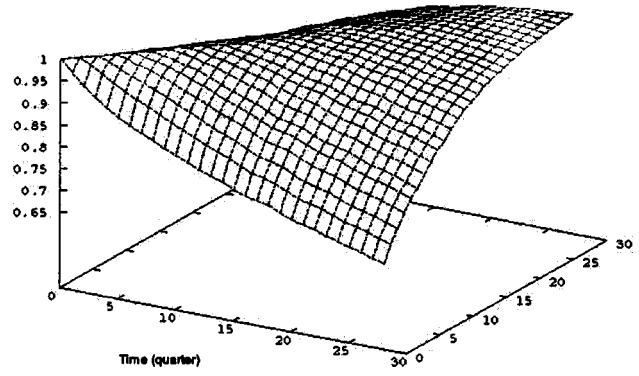

FIG. 3. The normalized correlation structure $\langle \delta f(t,\theta)\delta f(t,\theta')\rangle / \sqrt{\langle \delta f^2(t,\theta)\rangle}\sqrt{\langle \delta f^2(t,\theta')\rangle}$ observed in the market.

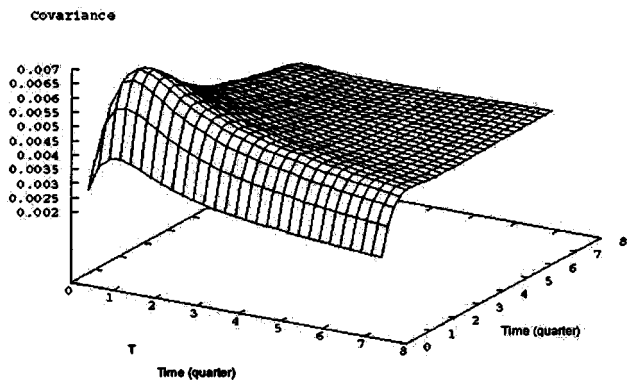

FIG. 4. The covariance of changes of forward rates $\langle \delta f(t,\theta)\delta f(t,\theta')\rangle$ observed in the market.

$$S = -\frac{1}{2}\int_{t_0}^{t_1} dt \int_0^{\infty} d\theta \left[A^2 + \left(\frac{\partial A}{\partial \theta}\right)^2 \right]. \tag{17}$$

To obtain the predicted correlation structure from propagator (10), we take the limit $T_{\mathrm{FR}} \to \infty$ and obtain

$$D(\theta,\theta') = \mu e^{-\mu\theta_>} \cosh \mu\theta_< = \frac{\mu}{2}\left(e^{-\mu|\theta-\theta'|} + e^{-\mu(\theta+\theta')}\right). \tag{18}$$

The predicted correlation structure for this model can be found from this form of the propagator by normalization and from Eq. (12) and obtain

$$C^{(1)}_{\mathrm{QFT}}(\theta,\theta') = \sqrt{\frac{e^{-\mu\theta_>}\cosh\mu\theta_<}{e^{-\mu\theta_<}\cosh\mu\theta_>}}. \tag{19}$$

To estimate the parameter μ from market data, we use the Levenberg-Marquardt method from Press *et al.* [5] to fit the parameters to the observed correlation structure shown in Fig. 3. The fitting was done by minimizing the square of the error. The overall correlation was fitted by $\mu = 0.061\ \mathrm{yr}^{-1}$. To obtain the error bounds, the data was split into 346 datasets of 500 contiguous days of data each and the estimation done for each of the sets. The 90% confidence interval for this dataset is (0.057, 0.075). Note that the confidence interval is asymmetric from the overall best fit due to the nonlinear dependence of correlation (19) on μ. The root mean square for the correlation for the best fit value is 4.23%, which shows that the model's prediction for the correlation structure is not very good.

The main problem, as can be seen from a comparison between the prediction for the best fit μ in Fig. 5 and the actual correlation structure in Fig. 3, is that the prediction is largely independent of the actual value of θ and largely determined by $|\theta - \theta'|$, which is not the case in reality. The correlation rapidly increases as θ increases in reality.

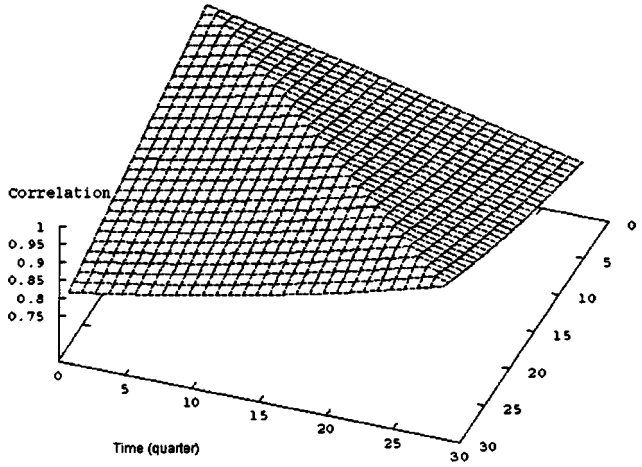

FIG. 5. Fitted correlation $C^{(1)}_{\mathrm{QFT}}$ for the constant rigidity model.

VIII. FIELD THEORY WITH CONSTRAINED SPOT RATE

One clear fact we notice from the covariance of the changes in the forward rates in Fig. 4 is that the covariance falls rapidly as $\theta \to 0$. This observation leads one to a model where $A(t,0)$ is constrained to follow a normal distribution with variance a. The mean of $A(t,0)$ can be fixed at any value, but will cause a corresponding change in $\alpha(0)$ which makes the mean value irrelevant. For calculational purposes it is easiest to assume that it remains at zero. This constraint can be implemented by modification of the action to

$$e^{S_{\mathrm{constrained}}} = \int_{-\infty}^{\infty} d\xi e^{i\xi A(t,0)} e^{-a^2\xi^2/2} e^{S}, \tag{20}$$

where S is the action specified in Eq. (17). The propagator $D(\theta,\theta')$ for this model is given by

$$D(\theta,\theta') = \mu e^{-\mu\theta_>}\left(\cosh\mu\theta_< - \frac{\mu e^{-\mu\theta_<}}{\mu+a}\right). \tag{21}$$

After normalizing, we see that the prediction for the correlation structure is given by

$$C^{(2)}_{\mathrm{QFT}}(\theta,\theta') = \left(\frac{e^{-\mu\theta_>}\left(\cosh\mu\theta_< - \frac{\mu e^{-\mu\theta_<}}{\mu+a}\right)}{e^{-\mu\theta_<}\left(\cosh\mu\theta_> - \frac{\mu e^{-\mu\theta_>}}{\mu+a}\right)}\right)^{1/2}. \tag{22}$$

We can see that the free parameters are μ and a. Further, it will be seen that it is easier to consider the ratio a/μ^2 as it is dimensionless. The results of the Levenberg-Marquardt method show that the fitted values of μ and a were very small, of the order of 10^{-7} yr^{-1} for μ and 10^{-13} yr^{-2} for a, both being very unstable but the ratio a/μ^2 was stable with a value in the range (6.7,10.7) with an overall best fit of 9.4. The most reasonable explanation for this behavior is that the ratio a/μ^2 determines the behavior of Eq. (22) for small μ and it is this region of the parameter space which gives a correlation structure closest to the empirically observed one.

The correlator for the constrained propagator looks very similar to Fig. 5, except that the behavior at large θ is slightly better when the constraint is put in. The root mean square error was 3.35% which again means that the fit was not very good, though significantly better than if the constraint was not applied. It must be recognized that the constraint introduces one extra free parameter which should improve the best fit. Hence, we see that this model, while again performing better than HJM, is still not very accurate. While the results are not very good, they do represent a reasonable first approximation and are still significantly better than the one-factor HJM model.

IX. FIELD THEORY MODEL WITH MATURITY DEPENDENT RIGIDITY $\mu = \mu(\theta)$

Another way to get a correlation structure that depends directly on the values of θ and θ' in a significant way and not only on their difference is to make μ a function of θ. This has a direct physical meaning as it means that if we imagine the forward rate curve as a string, its rigidity increases as maturity increases, making A for larger θ more strongly correlated if μ decreases as a function of θ.

We choose an exactly solvable function $\mu = \mu_0/(1+\lambda\theta)$; it declines to zero as θ becomes large, as is expected from the observed covariance in Fig. 4, and contains the constant μ case as a limit. The action is given by

$$S = -\frac{1}{2}\int_{t_0}^{t_1} dt \int_0^{\infty} d\theta \left[A^2 + \left(\frac{1+\lambda\theta}{\mu_0}\frac{\partial A}{\partial\theta}\right)^2\right]. \tag{23}$$

This is still a quadratic action and can be put into a quadratic form by performing integration by parts and setting the boundary term to zero, since we are assuming Neumann boundary conditions. The inverse (Green's function) of the quadratic operator, namely, the propagator for this action, is found to be

$$\begin{aligned} D(\theta,\theta';T_{\mathrm{FR}}) = {} & \frac{\mu_0^2\alpha}{2\lambda\alpha(\alpha+1/2)[1-(1+\lambda T_{\mathrm{FR}})^{-2\alpha}]} \\ & \times\left(\frac{\alpha+1/2}{\alpha-1/2}(1+\lambda T_{\mathrm{FR}})^{-2\alpha}(1+\lambda\theta_>)^{\alpha-1/2} \right. \\ & \left. +(1+\lambda\theta_>)^{-\alpha-1/2}\right) \\ & \times\left(\frac{\alpha+1/2}{\alpha-1/2}(1+\lambda T_{\mathrm{FR}})^{-2\alpha}(1+\lambda\theta_<)^{\alpha-1/2} \right. \\ & \left. +(1+\lambda\theta_<)^{-\alpha-1/2}\right), \end{aligned} \tag{24}$$

where $\alpha = \sqrt{\frac{1}{4}+\mu_0^2/4\lambda^2}$.

Note that we have explicitly put the bound of T_{FR} on the θ variable. The reason for this is that the limits have to be taken carefully in order to compare this model to the HJM in the limit $\mu_0 \to 0$ and to the constant rigidity field theory model when $\lambda \to 0$.

Let us first consider the limit $\lambda \to 0$. Note

$$\alpha = \left(\frac{1}{4} + \frac{\mu_0^2}{\lambda^2}\right)^{1/2} \sim \frac{\mu_0}{\lambda}\left(1 + \frac{\lambda^2}{4\mu_0^2}\right)^{1/2} \sim \frac{\mu_0}{\lambda}. \tag{25}$$

Therefore, we have

$$(1+\lambda\theta)^{-\alpha-1/2} = [(1+\lambda\theta)^{1/\lambda}]^{-\mu_0}(1+\lambda\theta)^{-1/2} \sim e^{-\mu_0\theta}. \tag{26}$$

Similarly $(1+\lambda\theta)^{\alpha-1/2} \sim e^{\mu_0\theta}$, $(1+\lambda\theta)^{-\alpha-1/2} \sim e^{-\mu_0\theta}$, and $(1+\lambda T_{\mathrm{FR}})^{-2\alpha} \sim e^{-2\mu_0 T_{\mathrm{FR}}}$. Putting all these limits into Eq. (24) and performing some straightforward simplifications, we see that Eq. (24) becomes equal to Eq. (10) in the limit $\lambda \to 0$. In taking this limit, we did not have any difficulty with T_{FR}. However, for the HJM limit, we will see that the limit $T_{\mathrm{FR}} \to \infty$ has to be taken only after the limit $\mu_0 \to 0$ has been taken.

Let us now consider the limit $\mu_0 \to 0$. In this limit, $\alpha \sim \frac{1}{2} + \mu_0^2/\lambda^2$. Hence, only one term in Eq. (24) survives as all the others are multiplied by $\alpha - 1/2$. This surviving term can be evaluated as

$$\frac{\mu_0^2}{2\lambda}\frac{(\alpha+1/2)^2}{\alpha-1/2}\frac{1}{1-(1+\lambda T_{\mathrm{FR}})^{-1}}(1+\lambda T_{\mathrm{FR}})^{-1} = \frac{\mu_0^2}{2\lambda}\frac{2\lambda^2}{\mu_0^2}\frac{(1+\lambda T_{\mathrm{FR}})}{\lambda T_{\mathrm{FR}}}\frac{1}{(1+\lambda T_{\mathrm{FR}})} = \frac{1}{T_{\mathrm{FR}}}. \tag{27}$$

The terms $(1+\lambda\theta_>)^{\alpha-1/2}$ and $(1+\lambda\theta_<)^{\alpha-1/2}$ obviously go to 1 in this limit and so were not included in the calculation above. This result can be seen to be equivalent to the HJM propagator after normalization. If the limit $T_{\mathrm{FR}} \to \infty$ is taken first, then the propagator becomes

$$D(\theta,\theta') = \frac{\mu_0^2(\alpha-1/2)}{2\lambda\alpha(\alpha+1/2)}(1+\lambda\theta_>)^{-\alpha-1/2} \times \left(\frac{\alpha+1/2}{\alpha-1/2}(1+\lambda\theta_<)^{\alpha-1/2} + (1+\lambda\theta_<)^{-\alpha-1/2}\right), \tag{28}$$

which exhibits a θ dependence in the limit $\mu_0 \to 0$. Hence, this cannot be made equivalent to HJM if the limits are taken in the wrong order. This problem is not present in the constant rigidity model.

For comparison with market data, we still take the limit $T_{\mathrm{FR}} \to \infty$ as the model is still directly related to the field theory model. The predicted correlation structure for this model is then given by

$$C^{(3)}_{\mathrm{QFT}}(\theta,\theta') = \left(\frac{(\alpha+1/2)(1+\lambda\theta_<)^{2\alpha}+\alpha-1/2}{(\alpha+1/2)(1+\lambda\theta_>)^{2\alpha}+\alpha-1/2}\right)^{1/2}. \tag{29}$$

We fitted the parameters μ_0 and λ to the correlation structure observed in the market in a similar manner as for the field theory model and obtained the results $\mu_0 = 1.2\times10^{-5}\ \mathrm{yr}^{-1}$ and $\lambda = 0.108\ \mathrm{yr}^{-1}$. The root mean square error in the correlation was 3.35%. On performing the error analysis for the parameters, it was found that μ_0 is very unstable but always very small (less than $10^{-2}\ \mathrm{yr}^{-1}$), while the 90% confidence interval for λ is (0.099, 0.149).

The relatively high value for λ seems to show that the falloff of the rigidity parameter $\mu = \mu_0/(1+\lambda\theta)$ is fairly rapid. The error reduces from 4.23% to 3.35%, but an extra parameter λ has to be added and the model becomes considerably more complicated. Further, we seem to be in the region of very small μ_0, which does not behave well in the HJM limit. In fact, the correlation structure in this limit is given by

$$\lim_{\mu_0\to 0} C^{(3)}_{\mathrm{QFT}}(\theta,\theta') = \left(\frac{1+\lambda\theta_<}{1+\lambda\theta_>}\right)^{1/2}. \tag{30}$$

Due to the very small value of μ_0 for the fitted function, this is a very good approximation for the fit. The obtained fit for the correlation function yields a propagator very similar to the one given for the constant rigidity in Fig. 5.

The limited improvement, the relatively complicated form of the correlation, and the near zero μ_0 problem prompted us to consider a different way of approaching the problem, which presented a much more satisfactory solution. This model is described in the following section.

X. FIELD THEORY MODEL WITH $f(t,z(\theta))$

To see where we might make an improvement, we note that the predicted correlation structure with the field theory model is largely defined by the $e^{-\mu|\theta_> - \theta_<|}$ term, which means that the correlation does not depend explicitly on the times $\theta_>$ and $\theta_<$.[2] However, we see immediately from Fig. 3 that the correlation increases significantly as we increase $\theta_>$ and $\theta_>$. This is intuitively reasonable as market participants are likely to treat the difference between 10 and 15 yr into the future quite differently from the difference between now and five years. In other words, there is good reason to expect $\lim_{\theta_< \to \infty} D(\theta_>, \theta_<) = 1$.[3] This is not satisfied by the constant rigidity models or by the varying rigidity model (if the limit $T_{\mathrm{FR}} \to \infty$ is taken). For the latter model this is slightly surprising, since $\mu \to 0$ as $\theta \to \infty$ and we might expect that for large θ the varying rigidity model should go into the HJM model limit ($D=1$). However, this does not happen, as previously discussed, since we have taken the limit $T_{\mathrm{FR}} \to \infty$.

Note that introducing the metric $z(x\text{-}t)$ is different from giving a maturity dependence to the rigidity function $\mu(\theta)$. To see this, we write the action with the rigidity function $\mu(\theta)$ as

$$S_{\mathrm{old}} = -\frac{1}{2}\int_{t_0}^{t_1} dt \int_0^\infty d\theta \left[A^2 + \frac{1}{\mu_0^2}\left(\frac{\partial A}{\partial z}\right)^2\right], \tag{31}$$

[2]There is another term of the form $e^{-\mu(\theta_> + \theta_<)}$, but this has only a small effect on the correlation structure.

[3]Obviously, $\theta_< \to \infty$ automatically implies $\theta_> \to \infty$.

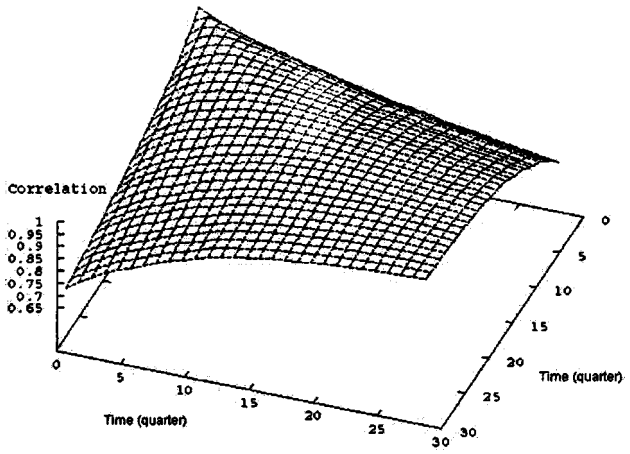

FIG. 6. Observed normalized correlation structure $\langle \delta f(t,\theta)\delta f(t,\theta')\rangle / \sqrt{\langle \delta f^2(t,\theta)\rangle}\sqrt{\langle \delta f^2(t,\theta')\rangle}$: a different view.

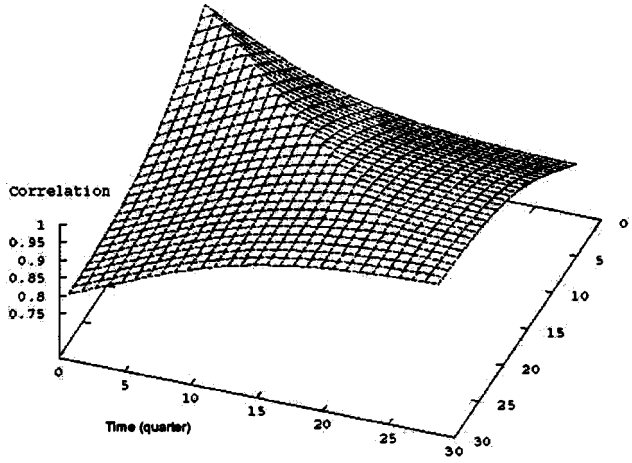

FIG. 7. Fitted correlation for the model with metric $g(\theta) = \tanh \beta\theta$.

where the functional variation of μ with θ has been absorbed into the variable $z=g(\theta)$ (where g is invertible) so that μ_0 above is a constant. With a change of variables we get the action as

$$S_{\text{old}} = -\frac{1}{2}\int_{t_0}^{t_1} dt \int_{g(0)}^{g(\infty)} dz\, h'(z)\left[A^2 + \frac{1}{\mu_0^2}\left(\frac{\partial A}{\partial z}\right)^2\right], \quad (32)$$

where $h=g^{-1}$. With the introduction of the metric, we obtain the action

$$S_{\text{new}} = -\frac{1}{2}\int_{t_0}^{t_1} dt \int_{g(0)}^{g(\infty)} dz\left[A^2 + \frac{1}{\mu_0^2}\left(\frac{\partial A}{\partial z}\right)^2\right] \neq S_{\text{old}}. \quad (33)$$

The Green's functions for S_{new} should be solved using the z variables, and as expected the solution is given by $D(z,z') = \frac{1}{2}[\exp -\mu_0|z-z'| + \exp -\mu_0(z+z')]$. It can be shown that the martingale condition is satisfied with the Green's function given by $D(z,z')$.

Bearing in mind the condition that, at large θ, the correlations should be close to 1, we choose a metric that satisfies the property $g(\theta)=\tanh \beta\theta$. We use this form of the metric to fit the correlation structure and obtain the result that $\mu = 0.48\ \text{yr}^{-1}$ and $\beta = 0.32\ \text{yr}^{-1}$ with a root mean square error of only 2.46%. Both parameters are also stable when the error analysis for the parameters is carried out. The 90% confidence interval for μ is (0.45, 0.58) and that for β is (0.22, 0.33). Hence, we see that even the parameter estimation for this model is more robust as the parameters are at least stable. Further, the shape of the fitted function is clearly closer to the observed one, as can be seen from Figs. 6, 5, and 7. The error that remains is largely confined to the correlation between the spot rate and other forward rates, which is not too surprising since the spot rate behaves very differently from the other forward rates.

We emphasize here that the coordinate $z(\theta)$ involves a fundamentally new way of considering the interest rate models.

TABLE I. Summary of field theory models of the forward rates [μ, $\sqrt{a}$, λ, β are all measured in units of $(\text{yr})^{-1}$].

Model	$C(\theta,\theta')$	μ	a/μ^2	λ	β	$\sqrt{\text{Error}}$
Constant rigidity μ	$\left(\frac{e^{-\mu\theta_>}\cosh \mu\theta_<}{e^{-\mu\theta_<}\cosh \mu\theta_>}\right)^{1/2}$	0.06				4.23%
Constrained spot rate	$\left(\frac{e^{-\mu\theta_>}\left(\cosh \mu\theta_< - \frac{\mu e^{-\mu\theta_<}}{\mu+a}\right)}{e^{-\mu\theta_<}\left(\cosh \mu\theta_> - \frac{\mu e^{-\mu\theta_>}}{\mu+a}\right)}\right)^{1/2}$		9.4			3.54%
$\mu = \frac{\mu_0}{1+\lambda\theta}; \alpha = \left(\frac{1}{4}+\frac{\mu_0^2}{4\lambda^2}\right)^{1/2}$	$\left(\frac{(\alpha+1/2)(1+\lambda\theta_<)^{2\alpha}+\alpha-1/2}{(\alpha+1/2)(1+\lambda\theta_>)^{2\alpha}+\alpha-1/2}\right)^{1/2}$	0.011		0.1		3.35%
$z=\tanh(\beta\theta)$	$\left(\frac{e^{-\mu z_>}\cosh \mu z_<}{e^{-\mu z_<}\cosh \mu z_>}\right)^{1/2}$	0.48			0.31	2.46%

XI. SUMMARY

We summarize below in Table I the results for the various field theory models that we have analyzed.

Most models in finance for the forward rates are generalizations of the HJM model. Our empirical study shows that the one-factor HJM model has too little freedom and that the two-factor HJM model has too much freedom. While retaining this HJM framework, the field theory generalization allows us to reasonably match the observed behavior of the forward rates and avoid the pitfalls of the HJM model. An important advantage of Gaussian field theory models is that, unlike the common practice in the HJM model, one can obtain the volatility of the forward rates directly from market data.

One can further refine the field theory model by using empirical data as a guide and possibly include effects of market psychology in the model, and has been done in [6].

ACKNOWLEDGMENTS

We would like to thank Jean-Philippe Bouchaud and Science and Finance for kindly providing us with the data used for this study. One of us (B.E.B.) would like to thank Cui Liang for verifying the figures.

[1] B. E. Baaquie, Phys. Rev. E **64**, 016121 (2001).
[2] B. E. Baaquie, Phys. Rev. E **65**, 056122 (2002).
[3] B. E. Baaquie, *Quantum Finance* (Cambridge University Press, Cambridge, 2004).
[4] A. Matacz and J.-P. Bouchaud, Int. J. Theor. Appl. Finance **3**, 703 (2000).
[5] W. H. Press, S. A. Teukolsky, W. T. Vettering, and B. P. Flannery, *Numerical Recipes in C: The Art of Scientific Computing* (Cambridge University Press, Cambridge, 1995).
[6] B.E. Baaquie and J.-P. Bouchaud (unpublished).